indefinite integral (no bounds)

U-Substitution – More Examples

ex. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \sin\sqrt{x} \cdot \underbrace{\frac{1}{\sqrt{x}}\,dx}$

$u = \sqrt{x}$

$2\,du = \frac{1}{2\sqrt{x}}\,dx$

$= 2\int \sin u \; du$

$= -2\cos u + C$

$= \boxed{-2\cos\sqrt{x} + C}$

$(\cos x)' = -\sin x$

$-(\cos x)' = \sin x \Rightarrow \int \sin x\,dx = -\cos x + C$

ex. $\int \frac{\arctan x}{1+x^2}\,dx = \int \arctan x \cdot \underbrace{\frac{1}{1+x^2}\,dx}$

$(\arctan x)' = \frac{1}{1+x^2}$

$= \int u \quad du$:)

$u = \arctan x$

$du = \frac{1}{1+x^2}\,dx$

$= \frac{u^2}{2} + C$

$= \boxed{\frac{1}{2}(\arctan x)^2 + C}$

Do: $(\cot x)' = -\csc^2 x$

Do: $\int \sqrt{\cot x}\,\underline{\csc^2 x\,dx}$

$\to u = \cot x$

$du = -\csc^2 x\,dx \Rightarrow -du = \csc^2 x\,dx$

$= -\int \sqrt{u}\;du$

$= -\int u^{1/2}\,du$

$= -\frac{2}{3}u^{3/2} + C$

$= \boxed{-\frac{2}{3}(\cot x)^{3/2} + C}$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

Definite Integrals with u-substitution

Recall FTC Part 2: $\int_a^b f(x)\,dx = F(b) - F(a)$

Do: $\int_0^7 \sqrt{4+3x}\,dx$

$u = 4+3x \longrightarrow x=0 \quad a = 4+3(0) = 4$

$\frac{1}{3}du = \frac{\cancel{3}}{\cancel{3}}dx \qquad x=7 \quad b = 4+21 = 25$

x-bounds

$= \int_4^{25} \sqrt{u}\ \frac{1}{3}du$

get u-bounds

$= \frac{1}{3}\cdot\frac{2}{3}u^{3/2}\Big|_4^{25}$

$= \frac{2}{9}\left(25^{3/2} - 4^{3/2}\right)$

$(25^{1/2})^3 = \sqrt{25}^3 = 5^3 = 125$

$(4^{1/2})^3 = 2^3 = 8$

$= \frac{2}{9}(125-8)$

$= \frac{2}{9}\cdot 117 = \frac{234}{9}$

$117 + 117 = 234$

$117 \div 9 = 13$

$= \frac{2\cdot\cancel{9}\cdot 13}{\cancel{9}} = \boxed{26}$

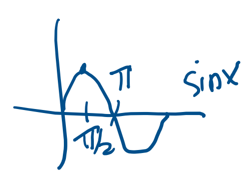
ex. $\int_0^{\pi/2} \cos x \cdot \sin(\sin x)\,dx$

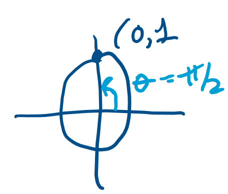
$u = \sin x \longrightarrow x_a = 0 \quad u_a = \sin 0 = 0$

$du = \cos x\,dx \qquad x_b = \frac{\pi}{2} \quad u_b = \sin\frac{\pi}{2} = 1$

$= \int_0^{\pi/2} \sin(\sin x)\cdot \cos x\,dx$

$= \int_0^1 \sin u\ du$

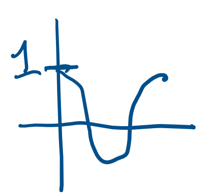




$= -\cos u\Big|_0^1$

$= -(\cos 1 - \cos 0)$

F(b) - F(a)



$= -\cos 1 + 1$

$= \boxed{1 - \cos 1}$ cannot be evaluated w/o a calculator

Symmetry Review given $f(x)$

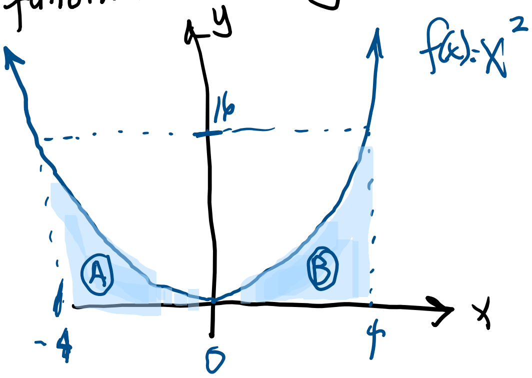
Even Functions

$f(x)$ is even when $f(-x) = f(+x)$

ex. even functions : $x^2 = f(x)$, $\cos x$

$f(-x) = (-x)^2 = x^2 = f(x)$

graphs of even functions have <u>y-axis</u> symmetry



$$\int_{-4}^{4} x^2\,dx = \underbrace{\int_{-4}^{0} x^2\,dx}_{\text{area of (A)}} + \underbrace{\int_{0}^{4} x^2\,dx}_{\text{area of (B)}}$$

$$= 2\int_{0}^{4} x^2\,dx$$

double area to get $\int_{-4}^{4} x^2\,dx$

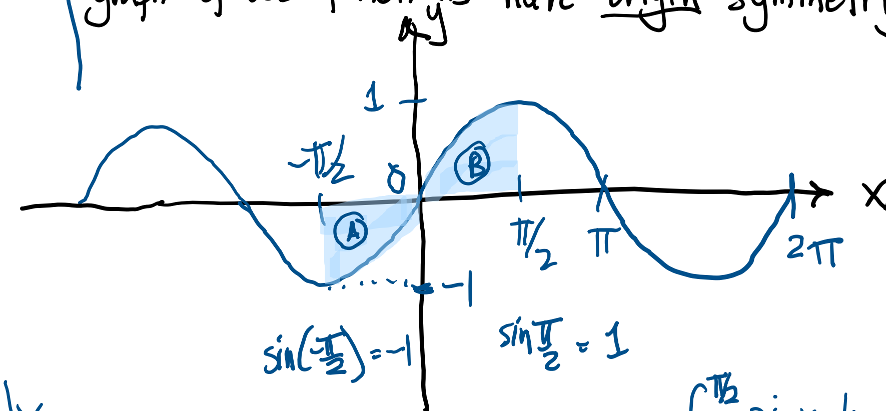
Odd Functions

$f(x)$ is odd when $f(-x) = -f(+x)$

ex. odd functions : $x^3 = f(x)$, $\sin x$, $\tan x$

$f(-x) = (-x)^3 = (-x)(-x)(-x) = x^2(-x) = -x^3 = -f(x)$

graphs of odd functions have <u>origin</u> symmetry



$$\int_{-\pi/2}^{\pi/2} \sin x\,dx$$

$$= \int_{-\pi/2}^{0} \sin x\,dx + \int_{0}^{\pi/2} \sin x\,dx$$

$$= \text{area (A)} + \text{area (B)} = \boxed{0}$$

Do: Determine if the following functions are even or odd:

$f(x) = x^4 + 1$

$f(-x) = (-x)^4 + 1$

$= x^4 + 1 = f(x)$

$\therefore f(x)$ is even

$f(x) = \dfrac{\tan x}{x^2}$ is odd

$f(-x) = \dfrac{\tan(-x)}{(-x)^2}$ ← odd ← even

$= \dfrac{-\tan x}{x^2}$

$= -\left(\dfrac{\tan x}{x^2}\right)$

$= -f(x)$

Integrating Symmetric Functions

Rules: suppose $f(x)$ is continuous on $[-a, a]$

if f is even: $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

if f is odd: $\int_{-a}^{a} f(x)\,dx = 0$

ex. $\int_{-1}^{1}\left(x^4+1\right)dx$

saw this was an even function on previous page

$$= 2\int_0^1 (x^4+1)\,dx$$

$$= 2\left(\frac{x^5}{5}+x\right)\Big|_0^1$$

$$= 2\left(\underbrace{\frac{1^5}{5}+1}_{F(b)} \;-\; \underbrace{(0+0)}_{F(a)}\right)$$

$$= 2\left(\frac{1}{5}+\frac{5}{5}\right)$$

$$= 2\left(\frac{6}{5}\right) = \boxed{\frac{12}{5}}$$

we showed previously that it's an odd function

ex. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan x}{x^2}\,dx = \boxed{0}$

Building on Taking the Derivative of an Integral

Recall FTC Part 1: If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

constant

Do: $\frac{d}{dx}\int_1^x t^2\,dt = \boxed{x^2} \cdot x^1$

Why can we "ignore" the constant?

re-visit: $\frac{d}{dx}\underbrace{\int_1^x t^2\,dt}$ integrate

$= \frac{d}{dx}\left[\frac{t^3}{3}\Big|_1^x\right]$ next $F(b) - F(a)$

$= \frac{d}{dx}\left(\frac{x^3}{3} - \frac{1^3}{3}\right)$

$= \frac{d}{dx}\left(\frac{1}{3}\cdot x^3\right) - \frac{d}{dx}\left(\frac{1}{3}\right)$

$= \frac{1}{\cancel{3}}\cdot \cancel{3}x^2 - 0$

$= \boxed{x^2}$